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The complete Whittaker theorem for two-dimensional integrable systems and its application

A Ankiewicz and C Pask
Department of Applied Mathematics, Research School of Physical Sciences, Australian National University, Canberra, ACT 2600, Australia

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Abstract. We prove, for a particle moving in a plane under the influence of a conservative force, that when the motion is constrained by a 'second' invariant quadratic in the velocities, then the potential allows separability of the Hamilton–Jacobi equation in rectangular, polar, elliptical cylinder or parabolic cylinder coordinates. This link shows the intimate connection between quadratic invariants and the two-dimensional Hamilton–Jacobi equation. We give examples of the utility of parabolic cylinder coordinates in cases of recent study.

1. Introduction

The motion of a particle moving in a plane under the action of conservative forces is an old problem, but recent developments have revealed a rich and intricate behaviour in all but the most simple cases (Berry 1978, Helleman 1980). It is now known that most cases are not integrable and that an irregular phase space motion usually occurs. There is no general criterion for selecting those potentials which give rise to a particular type of behaviour, but there are certain classic results covering whole classes of potentials.

The solution of the dynamical problem is possible if we have a separable Hamilton–Jacobi equation (Landau and Lifshitz 1969). In that case there is a constant of motion (the separation constant) in addition to the energy, and the resulting motion is regular. The work of Stäckel (1893) and Eisenhart (1948) shows that for motion in a plane there are specific forms of the potential leading to separable Hamilton–Jacobi equations in orthogonal coordinate systems only for the four cases of rectangular, polar, parabolic elliptical cylinder coordinates.

In his classic book, Whittaker (1944) considers the problem in the reverse way: assuming an invariant of motion other than energy, which potentials are possible? Whittaker gave a theorem for the special case of an invariant which is quadratic in the velocities (Whittaker 1944, § 152).

In this paper we point out that Whittaker's theorem is incorrect in the way in which he stated it, and that, in its complete form, it is actually complementary to the Stäckel (1893) and Eisenhart (1948) result. We also explain how the correct theorem is relevant to several recent studies of dynamical systems.

2. The complete Whittaker theorem

'The only cases of the motion of a particle in a plane, under the action of conservative forces, which possess an integral quadratic in the velocities other than the energy, are those in which the potential is such that the Hamilton–Jacobi equation is separable in cartesian, polar, elliptical or parabolic cylinder coordinates'.

2.1. *The proof*

Whittaker (1944) set up the mechanism for the general proof by assuming an invariant quadratic in the velocities and finding the partial differential equation satisfied by the potential V. However, by assuming a constant never to be zero, he eliminated a whole class of potentials and gave an incomplete result. Whittaker mentioned cases for cartesian and polar cordinates and gave the correct result for elliptical cylinder coordinates, but he ignored the case where the potential $V(x, y)$ satisfies

$$(\partial^2 V/\partial y^2 - \partial^2 V/\partial x^2)(-b'y - bx + c_1) + 2(\partial^2 V/\partial x\,\partial y)(by - b'x + c_2) + 3b\,\partial V/\partial x - 3b'\,\partial V/\partial y = 0 \tag{1}$$

where b, b', c_1 and c_2 are constants. Potentials satisfying (1) have the required type of invariant; they lead to a separable Hamilton–Jacobi equation when a conversion to parabolic cylinder coordinates is made, as we now demonstrate.

We begin by making a shift in the origin and then a clockwise rotation of axes by $\tan^{-1}(b'/b)$ to give new coordinates (X, Y):

$$X = b(x - c_3) - b'(y - c_4), \qquad Y = b'(x - c_3) + b(y - c_4). \tag{2a, b}$$

If the constants c_3 and c_4 are chosen to be

$$c_3 = (bc_1 + b'c_2)/(b^2 + b'^2), \qquad c_4 = (b'c_1 - bc_2)/(b^2 + b'^2), \tag{3}$$

then equation (1) is transformed into

$$X(\partial^2 V/\partial X^2 - \partial^2 V/\partial Y^2) + 2Y\,\partial^2 V/\partial X\,\partial Y + 3\,\partial V/\partial X = 0. \tag{4}$$

We next set

$$V = U/(\xi^2 + \eta^2) \tag{5}$$

where we are transforming to parabolic cylinder coordinates, so that

$$X = \xi\eta \qquad Y = \tfrac{1}{2}(\xi^2 - \eta^2). \tag{6a, b}$$

Equation (4) then reduces to

$$\partial^2 U/\partial\xi\,\partial\eta = 0 \tag{7}$$

with the solution

$$U = g_1(\xi) + g_2(\eta) \tag{8}$$

where g_1 and g_2 are arbitrary functions. The potential V is given by equations (5) and (8),

$$V = (g_1(\xi) + g_2(\eta))/(\xi^2 + \eta^2) \tag{9}$$

and we see that it has just the form required for separability of the Hamilton–Jacobi equation (Landau and Lifshitz 1969, § 48).

In the other case when an invariant quadratic in the velocities exists, Whittaker (1944) has shown that V can be expressed as

$$V = (\phi_1(\alpha) - \phi_2(\beta))/(\alpha^2 - \beta^2) \tag{10}$$

where the constant α and constant β curves are confocal ellipses and hyperbolae. The functions ϕ_1 and ϕ_2 in (10) are arbitrary. We point out that (10) also contains the two other coordinate systems to be considered: let the focal distance be f and then

$$V = h_1(r) + h_2(\psi)/r^2, \qquad f \to 0, \tag{11a}$$

$$V = k_1(x) + k_2(y), \qquad f \to \infty. \tag{11b}$$

In equation (10), (r, ψ) are polar coordinates, (x, y) are cartesian rectangular coordinates and h_1, h_2, k_1, k_2 are arbitrary functions. The above expressions for V, (10), (11a) and (11b), all lead to separable Hamilton–Jacobi equations (Landau and Lifshitz 1969, § 48), and so the complete Whittaker theorem is established.

3. Applications

The classes of potentials discussed have wide application in mechanics, optics and other branches of physics. The potential in a mechanical system corresponds to the refractive index profile in a multimode optical fibre (Ankiewicz and Pask 1983). For example, refractive index distributions expressible in the form of equation (10) have been used to predict and explain the observed elliptic and hyperbolic caustic curves in slightly non-circular graded index optical fibres (Ankiewicz 1979). The case of motion in the presence of two gravitational centres is the classic case for use of elliptic coordinates (Whittaker 1944, § 53). However, we mainly wish to discuss the class which Whittaker overlooked, i.e. potentials separable in parabolic cylinder coordinates. This class has been used to solve optical fibre bending problems (Gambling and Matsumura 1977) and is vital for many applications of current interest in dynamics.

3.1. *Potentials and the Painlevé property*

The 'weak Painlevé property' has been proposed as a criterion of integrability. By attempting to satisfy this property, a family of homogeneous potentials have been found (Ramani *et al* 1982):

$$V_n = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{2k} (2y)^{n-2k} \tag{12}$$

where $[n/2]$ is the largest integer $\leqslant n/2$, and $\binom{n-k}{k}$ is a binomial coefficient. We now show that this family leads to integrable Hamiltonians because each potential can be expressed in the form of equation (9) using parabolic cylinder coordinates ξ and η as defined in (6):

$$V_n = (\xi^{2n+2} + (-1)^n \eta^{2n+2})/(\xi^2 + \eta^2). \tag{13}$$

We prove that (13) is indeed the same as (12) by looking at the recurrence relations in each case, as well as the first two potentials in each set.

By writing the expansions and using the well known identity

$$\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$$

it can be seen that, for the potential of equation (12), we have

$$V_{n+1}=2yV_n+x^2V_{n-1} \tag{14}$$

with $V_0=1$ and $V_1=2y$.

If we multiply both sides of (13) by $(\xi^4-\eta^4)$ and rearrange, we find the recurrence relation

$$V_{n+1}=(\xi^2-\eta^2)V_n+\xi^2\eta^2V_{n-1} \tag{15}$$

with $V_0=1$ and $V_1=\xi^2-\eta^2$.

From the transformations used (equation (6)) it is clear that relations (14) and (15) are identical, as are V_0 and V_1 thus proving that V_n can be written as in equation (13) for all n.

We can now use the Hamilton–Jacobi formulation to find the second invariants explicitly for potentials as in (9). The Hamiltonian is

$$H=T+V=[1/(\xi^2+\eta^2)][\tfrac{1}{2}(p_\xi^2+p_\eta^2)+g_1(\xi)+g_2(\eta)]=E \tag{16}$$

where the energy E is conserved. The second invariant, C, can thus be written

$$C=\tfrac{1}{2}\dot{\xi}^2(\xi^2+\eta^2)^2+g_1(\xi)-E\xi^2 \tag{17}$$

$$=-\tfrac{1}{2}\dot{\eta}^2(\xi^2+\eta^2)^2-g_2(\eta)+E\eta^2. \tag{18}$$

By adding (17) and (18), and substituting in for E from (16), we obtain

$$C=(\xi^2+\eta^2)^{-1}(\eta^2g_1(\xi)-\xi^2g_2(\eta))-\tfrac{1}{2}(\xi^2+\eta^2)(\xi^2\dot{\eta}^2-\eta^2\dot{\xi}^2). \tag{19}$$

For the particular class we were considering,

$$g_1(\xi)=\xi^{2n+2}, \qquad g_2(\eta)=(-1)^n\eta^{2n+2}, \tag{20}$$

so

$$C_n=\xi^2\eta^2V_{n-1}-\tfrac{1}{2}(\xi^2+\eta^2)(\xi^2\dot{\eta}^2-\eta^2\dot{\xi}^2) \tag{21}$$

$$=x^2V_{n-1}+\dot{x}(x\dot{y}-\dot{x}y). \tag{22}$$

The form in cartesian coordinates (22) agrees with that given by Ramani *et al* (1982).

3.2. *Hénon–Heiles type potentials*

A case which has received much attention involves the integrable class of generalised Hénon–Heiles potentials:

$$V_{\mathrm{HH}}(x,y)=\tfrac{1}{2}(Ax^2+By^2)+D(x^2y+2y^3) \tag{23}$$

where A, B and D are constants. By writing down the derivatives $\partial V/\partial x$ etc, substituting them into equation (1) and equating like terms, we find coefficients b, b', c_1 and c_2. In this case $c_1=b'=0$, and $b(4A-B)+4Dc_2=0$. Thus the only coordinate change required is a shift of the y axis. From equations (2) and (3):

$$X=x \qquad \text{and} \qquad Y=y-c_4 \tag{24}$$

where

$$c_4 = -c_2/b = (4A-B)/4D. \tag{25}$$

Thus

$$\begin{aligned} V_{\mathrm{HH}}(X, Y) &= D(X^2Y+2Y^3)+\tfrac{1}{4}(6A-B)(X^2+4Y^2) \\ &\quad +(Y/8D)(4A-B)(12A-B)+(A/8D^2)(4A-B)^2 \qquad (26a) \\ &= (D/4)V_3+\tfrac{1}{4}(6A-B)V_2+[(4A-B)(12A-B)/16D]V_1 \\ &\quad +(A/8D^2)(4A-B)^2V_0 \qquad (26b) \end{aligned}$$

where the V_n are given by (12) and (13). Thus V_{HH} allows separation of the Hamilton–Jacobi equation in parabolic cylinder coordinates and a second invariant exists for all A, B, D. The invariant, in terms of the variables X and Y, is found by substituting the appropriate $g_1(\xi)$ and $g_2(\eta)$ into (19). The constant term in (26b) is immaterial. This gives

$$\begin{aligned} C &= X^2[\tfrac{1}{4}DX^2+DY^2+\tfrac{1}{2}(6A-B)Y+(1/16D)(4A-B)(12A-B)] \\ &\quad -\dot{X}(\dot{X}Y-X\dot{Y}). \qquad (27a) \end{aligned}$$

Using equation (24), we may go back to the (x, y) coordinates:

$$\begin{aligned} C_{\mathrm{HH}} &= -4DC = (B-4A+4Dy)\dot{x}^2-4Dx\dot{x}\dot{y}-x^2[D^2x^2+4D^2y^2 \\ &\quad +4ADy+A(4A-B)]. \qquad (27b) \end{aligned}$$

Equation (27b) is in agreement with the invariant found by other methods (Chang *et al* 1981, 1982, Grammaticos *et al* 1982).

3.3 Coupled quartic oscillators

Bountis *et al* (1982) consider potentials of the form

$$V_Q = \tfrac{1}{2}(Ax^2+By^2)+\tfrac{1}{4}(x^4+\sigma y^4+2\rho x^2y^2) \tag{28}$$

and find that the resulting particle motion is regular for $A = B$ when $\sigma = \rho = 1$ or $\sigma = 1$, $\rho = 3$. These two cases lead to separable problems in polar and rectangular coordinates. Following the sort of argument used above, we find two more integrable cases by converting to parabolic coordinates and requiring the conditions for separability, equation (9), to hold. The potential is expressed in terms of V_n, equations (12) and (13), as follows:

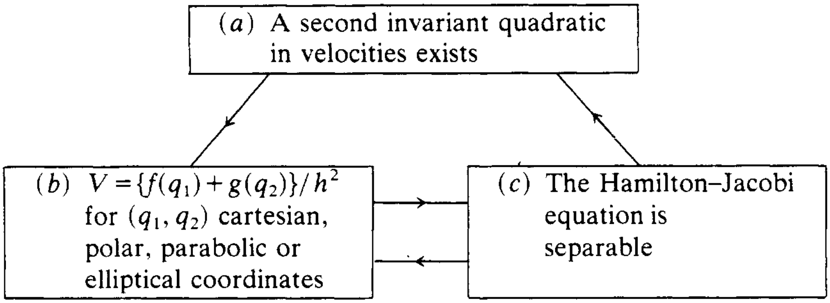
$$B=4A, \qquad \sigma=16, \qquad \rho=6: \qquad V_Q=\tfrac{1}{2}AV_2+\tfrac{1}{4}V_4 \tag{29a}$$

$$B=\tfrac{1}{4}A, \qquad \sigma=\tfrac{1}{16}, \qquad \rho=\tfrac{3}{8}: \qquad V_Q=\tfrac{1}{2}BV_2+\tfrac{1}{64}V_4. \tag{29b}$$

The invariants can be found from equation (19).

4. Conclusion

The following block diagram summarises the connections between statements made in this paper concerning motion by a particle in a plane under the influence of a potential V:



In (b), the functions f and g are arbitrary and h^2 is the square of the appropriate scale factor (Landau and Lifshitz 1969, § 48). The link (c) to (b) was established by Stäckel (1893) and Eisenhart (1948), and the link (c) to (a) is part of the theory of dynamics. The link (a) to (b) is provided by the complete Whittaker theorem as described in this paper.

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